We saw that a complex number $z = x + iy$ can be written in polar form as $z = r\cos\theta + r\sin\theta\, i = r(\cos\theta + i\sin\theta)$ with $r \geq 0$

Example: write $\sqrt{3} + i$ in polar form

Soln: $r = |\sqrt{3} + i| = \sqrt{\sqrt{3}^2 + 1^2} = 2 \Rightarrow \cos\theta = \sqrt{3}/2$ (why?)

$\sin\theta = \sqrt{3}/2$

$\Rightarrow \theta = \frac{\pi}{6}$ works so $\sqrt{3} + i = 2(\cos\frac{\pi}{6} + i\sin\pi/6)$

Polar form is useful for multiplying complex numbers.

Suppose we want to multiply

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ & } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then $z_1 z_2 = r_1 r_2 \big(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + i\big(\underbrace{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}_{\sin(\theta_1+\theta_2)}\big)\big)$

$$\Rightarrow \boxed{z_1 z_2 = (r_1 r_2)\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]}$$

Similarly: $z_1/z_2 = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

Can also show

$z = r(\cos\theta + i\sin\theta) \Rightarrow z^n = r^n(\cos n\theta + i\sin n\theta)$

De Moivre's Theorem

Ex: We saw that $\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\Rightarrow (\sqrt{3}+i)^{17} = 2^{17}\left(\cos\frac{17\pi}{6} + i\sin\frac{17\pi}{6}\right)$

$= 2^{17}\cdot\left(\underbrace{\cos 5\pi/6}_{-\sqrt{3}} + i\underbrace{\sin 5\pi/6}_{1}\right)$

$= 2^{17}(-\sqrt{3}+i)$

Roots of a complex number

the n^{th} root of z is a complex number w such that $w^n = z$

So, by de Moivre's formula, if $z = r(\cos\theta + i\sin\theta)$

and $w = s(\cos\phi + i\sin\phi)$

we have $s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$

So $s = r^{1/n}$ $\phi = \frac{\theta + 2\pi k}{n}$ where $k = 0, 1, \ldots, n-1$

So there are n-roots:

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

n^{th} roots of $r(\cos\theta + i\sin\theta)$

Example Find the 3rd roots of 1 (called the 3rd roots of unity)

Sol'n: $1 = 1(\cos 0 + i\sin 0)$

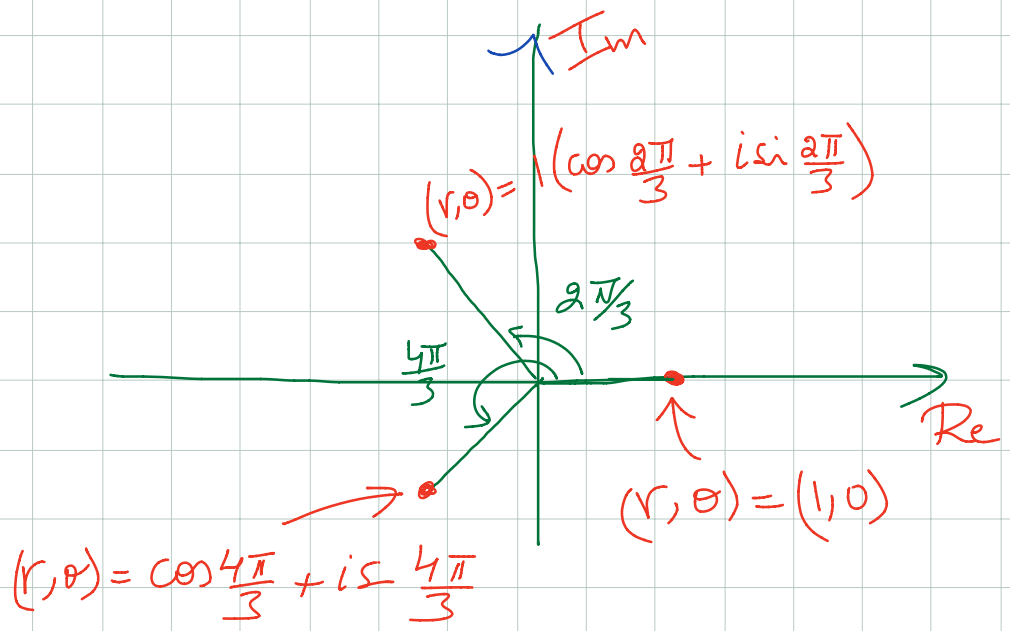
$$\Rightarrow w_k = 1^{1/3}\left(\cos\frac{0+2\pi k}{3} + i\sin\frac{0+2\pi k}{3}\right)$$

for $k=0,1,2$.

So $w_0 = 1$, $w_1 = -\frac{1}{2} + \sqrt{3}/2\, i$

$$w_2 = -\frac{1}{2} - \sqrt{3}/2\, i$$

Let's sketch these on a polar plot



Complex Exponentials

Define: $e^{a+\theta i} = e^a(\cos\theta + i\sin\theta)$

With this definition:

$$e^{a_1+\theta_1 i}\, e^{a_2+\theta_2 i} = e^{(a_1+a_2)+(\theta_1+\theta_2)i}$$

(so $e^{z_1+z_2} = e^{z_1}e^{z_2}$) where z_1 & z_2 are complex

Proof: Exercise

Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Consequences of Euler's formula:

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Very helpful for deriving trig identities

e.g. $\cos^2\theta = \cos\theta\cos\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$

$$= \left(e^{2i\theta} + \underbrace{e^{i(\theta-\theta)}}_{1} + \underbrace{e^{-i\theta+i\theta}}_{1} + e^{-2i\theta}\right)/4$$

$$= \frac{1}{2} + \frac{e^{2i\theta}+e^{-2i\theta}}{4} = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

—×—

Example/Exercise : Show that $e^{\pi i}+1=0$

Trigonometric Integrals (7.2)
& Integrals involving hyperbolic functions (7.4)

Example: Evaluate $\int \cos(3x)\sin(x)\,dx$

Method (1): See book

Method (2): Convert $\cos(3x)$ & $\sin(x)$ to complex exponentials and integrate

$$\Rightarrow \int \cos(3x)\sin(x)\,dx = \int \left(\frac{e^{i3x}+e^{-i3x}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx$$

$$= \int \frac{e^{i4x}-e^{i2x}+e^{-i2x}-e^{-i4x}}{4i}\,dx$$

$$= \int \frac{\left(e^{i4x}-e^{-i4x}\right)-\left(e^{i2x}-e^{-i2x}\right)}{4i}\,dx$$

Integrate normally

$$= \frac{1}{4i}\left[\left(\frac{e^{i4x}}{4i}-\frac{e^{-i4x}}{-4i}\right)-\left(\frac{e^{i2x}}{2i}-\frac{e^{-i2x}}{-2i}\right)\right]+C$$

$$= \frac{1}{4i}\left[\frac{e^{i4x}+e^{-i4x}}{2(2i)} - \frac{e^{i2x}+e^{-i2x}}{2i}\right] + C$$

($\cos(4x)$ over the first fraction; $\cos(2x)$ over the second)

$$= -\frac{1}{8}\cos 4x + \frac{1}{4}\cos(2x) + C$$

Now, let's integrate: $\int e^{2x}\sin x\,dx$

can do it by parts, or:

$$\int e^{2x}\sin x\,dx = \int e^{2x}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx$$

$$= \frac{1}{2i}\int e^{(2+i)x} - e^{(2-i)x}\,dx$$

$$= \frac{1}{2i}\left[\frac{e^{2x}e^{ix}}{2+i} - \frac{e^{2x}e^{-ix}}{2-i}\right] + C$$

$$= -\frac{e^{2x}}{2}\left[\frac{e^{ix}}{1-2i} + \frac{e^{-ix}}{1+2i}\right] + C$$

$$= -\frac{e^{2x}}{2}\left[\frac{(1+2i)e^{ix}}{5} + \frac{(1-2i)e^{-ix}}{5}\right] + C$$

$$= -\frac{e^{2x}}{5}\left[\underbrace{\frac{e^{ix}+e^{-ix}}{2}} - 2\cdot\underbrace{\left(\frac{e^{ix}-e^{-ix}}{2i}\right)}\right] + C$$

$$-\frac{e^{2x}}{5}(\cos x - 2\sin x) + C$$

Hyperbolic sine & cosine :

$$\cosh(x) := \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) := \frac{e^x - e^{-x}}{2}$$

Example: Evaluate $I = \int \cos^2(x) \sinh^2(x)\, dx$

Soln: • $\cos^2(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{e^{2ix} + 2e^0 + e^{-2ix}}{4}$

$$= \frac{1}{2}\left(\frac{e^{2ix} + e^{-2ix}}{2} + 1\right) = \frac{1}{2}(\cos 2x + 1)$$

• $\sinh^2(x) = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{2}\left(\frac{e^{2x} - 2e^0 + e^{-2x}}{2}\right)$

$$= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2} - 1\right) = \frac{1}{2}(\cosh(2x) - 1)$$

$$\Rightarrow \cos^2(x)\sinh^2(x) = \frac{1}{4}\left[\cos(2x)\cosh(2x) + \cosh(2x) - \cos(2x) - 1\right]$$

$\Rightarrow$

$$= \frac{1}{4}\cdot\frac{1}{2}\left(\frac{e^{2x}}{2} \mp \frac{e^{-2x}}{-2}\right) \qquad \sinh(2x)$$

$$I = \frac{1}{4}\int \cos(2x)\cosh(2x)\,dx + \frac{1}{4}\int \cosh(2x)\,dx$$

$$-\frac{1}{4}\int \cos(2x)\,dx - \int \frac{1}{4}\,dx$$

$$-\frac{1}{4}\frac{\sin 2x}{2} \qquad -\frac{1}{4}x$$

$$\Rightarrow I = \underbrace{\frac{1}{4}\int \cos(2x)\cosh(2x)\,dx}_{S} + \frac{1}{8}\sinh(2x) - \frac{1}{8}\sin(2x) - \frac{1}{4}x + C$$

$$S = \frac{1}{4}\int \cos(2x)e^{2x}\,dx + \frac{1}{4}\int \cos(2x)e^{-2x}\,dx$$

Exercise: Finish this problem

7.5 The Method of Partial Fractions

Use to integrate rational functions

$$F(x) = P(x)/Q(x)$$

$P(x)$: polynomial; $Q(x)$: polynomial

Some useful facts:

Fundamental Theorem of Algebra

Any nonconstant polynomial

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

(n = degree of $Q(x)$)

can be factored as

$$Q(x) = c(x-\beta_1)^{n_1}(x-\beta_2)^{n_2}\cdots(x-\beta_k)^{n_k}$$

c: complex constant; $\beta_1, \beta_2, \ldots, \beta_k$: complex

- The β_i's are called the zeros of $Q(x)$
- The n_i's are called the multiplicity of the β_i

Examples:

- $2x^2 - 8x + 6 = 2(x-3)(x-1)$
- $x^3 - 8 = (x-2)(x+1+i\sqrt{3})(x+1-i\sqrt{3})$
- $(x^2+1)^3 = (x+i)^3(x-i)^3$

$-i$ is a zero of multiplicity 3

Fundamental Theorem of Algebra: real factors

Any nonconstant polynomial with real coef's

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

can be factored as

$$Q(x) = c(x-r_1)^{m_1} \cdots (x-r_\ell)^{m_\ell}(x^2+b_1x+c_1)^{n_1} \cdots (x^2+b_k x+c_k)^{n_k}$$

r_1: real; r_ℓ: real; $x^2+b_1x+c_1$, ..., $x^2+b_kx+c_k$: irreducible quadratics with no real roots

Examples:

Evaluate $\int \frac{dx}{x^2-3x+2}\,dx$

Soln: ① Factor:

$$I = \int \frac{dx}{x^2-3x+2} = \int \frac{dx}{(x-2)(x-1)}$$

② write the partial fraction expansion

$$I = \int \frac{A}{x-2} + \frac{B}{x-1}\, dx$$

and find A & B:

$$\frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)} = \frac{1}{(x-2)(x-1)}$$

↑ common denom. ↑ we want this

need $A+B=0$

So $Ax+Bx-A-2B = 1 = (0x+1)$

$\Rightarrow A = -B$ & $-A-2B = 1$ } $B=-1$, $A=1$

need $-A-2B=1$

③ Sub & integrate

$$I = \int \frac{1}{x-2} + \frac{-1}{x-1}\, dx$$
$$= \ln|x-2| - \ln|x-1| + C$$

Example:

Evaluate $I = \int \frac{x^2+2}{(x-1)(2x-8)(x+2)}\,dx$

Soln: Since this is already factored, we go straight to the partial fraction expansion

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{A}{x-1} + \frac{B}{(2x-8)} + \frac{C}{x+2}$$

Now, find A, B, C: multiply by $(x-1)(2x-8)(x+2)$

$$\Rightarrow x^2+2 = A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8)$$

To find A: set $x=1$

$$\Rightarrow 1^2+2 = A(2(1)-8)(1+2) + B(0) + C(0)$$

$$3 = A(-6)(3) \Rightarrow A = -\frac{1}{6}$$

To find B: Set $x=4$ $(2x-8=0)$

$$\Rightarrow 4^2+2 = 0+B(3)(6)+0$$

$$\Rightarrow \boxed{B=1}$$

To find C: Set $x=-2$

$$\Rightarrow (-2)^2+2 = 0+0+C(-2-1)(2(-2)-8)$$

$$\Rightarrow C = \frac{6}{36} = \frac{1}{6}$$

So $$I = -\frac{1}{6}\int \frac{dx}{x-1} + \int \frac{dx}{2x-8} + \frac{1}{6}\int \frac{dx}{x+2}$$

$$= -\frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|x-4| + \frac{1}{6}\ln|x+2| + C$$

In both these examples: $\frac{P(x)}{Q(x)}$ was Proper that is $\text{degree}(P) < \text{degree}(Q)$

Example: (Here $\deg(P) \geq \deg(Q)$)

Evaluate $I = \int \frac{x^3+1}{x^2-4}\,dx$

Here, since $\deg P \geq \deg Q$

we have to do long division

$$\begin{array}{r} x \\ x^2-4 \overline{)\,x^3+1} \\ x^3-4x \\ \hline 4x+1 \end{array}$$

$$\Rightarrow \frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4}$$

So $I = \int \left(x + \frac{4x+1}{x^2-4}\right) dx$

$$= \int x\,dx + \int \frac{4x+1}{(x-2)(x+2)}\,dx$$

$$= x^2/2 + \int \frac{A}{x-2} + \frac{B}{x+2}\,dx$$

$$= x^2/2 + \int \frac{9/4}{x-2}\,dx + \int \frac{7/4}{x+2}\,dx$$

$$= x^2/2 + 9/4 \ln|x-2| + 7/4 \ln|x+2| + C$$

Example: Evaluate $I = \int \frac{3x-9}{(x-1)(x+2)^2}\,dx$

repeated linear factor

① $\frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

note these two terms

To find A, B, C:

$$3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

Plug in $x=-2 \Rightarrow C=5$

// $x=1 \Rightarrow A=-2/3$

to find B, plug some other number in: e.g. $x=2$

$$3(2)-9 = -\tfrac{2}{3}(2+2)^2 + B(2-1)(2+2) + 5(2-1)$$

(A ↙, C ↘)

$$\Rightarrow B = 2/3$$

So, $I = \int \frac{-2/3}{x-1}\,dx + \int \frac{2/3}{x+2}\,dx + \int \frac{5}{(x+2)^2}\,dx$

$$= -\tfrac{2}{3}\ln|x-1| + \tfrac{2}{3}\ln|x+2| - 5/(x+2) + C$$

Example

$$I = \int \frac{18}{(x+3)(x^2+9)}$$

↳ Irreducible quadratic factor

$$\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{(x^2+9)}$$

As usual: $18 = A(x^2+9) + (Bx+C)(x+3)$

set $x=-3 \Rightarrow A = 1$

So $18 = (x^2+9) + (Bx+C)(x+3)$

$\Rightarrow 18 = (B+1)x^2 + (C+3B)x + (9+3C)$

So $B = -1$ because $(B+1 = 0)$

$C = 3$ $\quad (C+3B = 0)$

$$\Rightarrow I = \int \frac{1}{x+3}\,dx + \int \frac{-x+3}{x^2+9}\,dx$$

$$= \underbrace{\int \frac{1}{x+3}\,dx}_{\ln|x+3|} - \underbrace{\int \frac{x}{x^2+9}\,dx}_{\frac{1}{2}\ln|x^2+9|} + \underbrace{\int \frac{3}{x^2+9}\,dx}_{\tan^{-1}(x/3)} + C$$

7.6) Improper Integrals

Example $\int_{-\infty}^{a} f(x)\,dx$ or $\int_{a}^{+\infty} f(x)\,dx$

or $\int_{-\infty}^{\infty} f(x)\,dx$

But what do they mean?

We define $\int_{-\infty}^{a} f(x)\,dx = \lim_{c \to -\infty} \int_{c}^{a} f(x)\,dx$

$\int_{a}^{\infty} f(x) = \lim_{b \to \infty} \int_{a}^{b} f(x)\,dx$

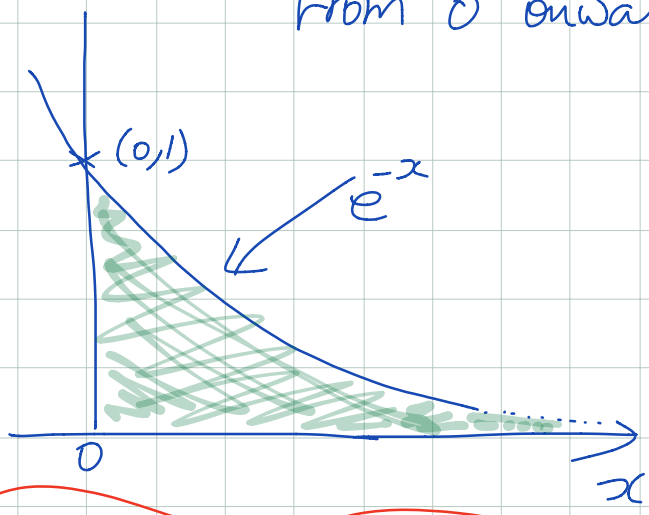
Example: $\int_{0}^{a} e^{-x}\,dx = -e^{-x}\Big|_{0}^{a}$

$= 1 - e^{-a}$

$\Rightarrow \lim_{a \to \infty} \int_{0}^{a} e^{-x}\,dx = \lim_{a \to \infty} (1 - e^{-a}) = 1 - 0$

$= 1$

This is the "area" under the curve e^{-x} from 0 onward.



Defn: The improper integral of $f(x)$ over $[a,\infty)$ is $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$ (if the limit exists).

If the limit exists we say the improper integral converges. Otherwise, we say it diverges.

Example Evaluate $\int_0^\infty e^{-x}\,dx$

Soln: $\int_0^\infty e^{-x}dx = \lim_{R\to\infty}\int_0^R e^{-x}dx$

$= \lim_{R\to\infty}\left(1-e^{-R}\right) = 1$

Example $\int_3^\infty \frac{1}{x^2}dx = \lim_{R\to\infty}\int_3^R x^{-2}dx$

$= \lim_{R\to\infty} -\frac{1}{x}\Big|_3^R = \lim_{R\to\infty} \frac{1}{3}-\frac{1}{R}$

$= 1/3$

Example: Show that $\int_{-\infty}^{-1}\frac{1}{x}dx$ diverges

$\lim_{R\to-\infty}\int_R^{-1}\frac{1}{x}dx = \lim_{R\to-\infty}\left[\ln|x|\right]_R^{-1}$

$= \lim_{R\to-\infty}\left(\underbrace{\ln|-1|}_{0} - \ln|R|\right)$

$= \lim_{R\to-\infty}\left(-\ln|R|\right) = -\infty$